Integration 3

1. $\int_{-1}^{1} \ln x \, dx$ (involve complex number)

Let $y = \ln x, x = e^y, \; dx = e^y dy$. When $x = -1, -1 = e^y, y = i\pi$, when $x = 1, y = 0$.

$$I = \int_{-1}^{1} \ln x \, dx = \int_{i\pi}^{0} y \, e^y dy = \int_{i\pi}^{0} y \, d(e^y) = [ye^y]_{i\pi}^{0} - \int_{i\pi}^{0} e^y \, dy = [ye^y - e^y]_{i\pi}^{0} = \underline{\underline{\mathbf{i\pi - 2}}}$$

2. Evaluate $L = \int_0^1 \sqrt{1 + 4x^2} \, dx$

Method 1

Let $x = \frac{1}{2} \tan\theta \Longrightarrow dx = \frac{1}{2} \sec^2\theta d\theta$

$$L = \int_0^{\tan^{-1}2} \sqrt{1 + 4\left(\frac{1}{2}\tan\theta\right)^2} \left(\frac{1}{2}\sec^2\theta d\theta\right) = \frac{1}{2}\int_0^{\tan^{-1}2} \sec^3\theta d\theta$$

Let $I = \int \sec^3\theta d\theta = \int \sec\theta d(\tan\theta) = \sec\theta\tan\theta - \int \tan\theta \, d(\sec\theta)$

$= \sec\theta\tan\theta - \int \tan\theta \, (\sec\theta\tan\theta \, d\theta) = \sec\theta\tan\theta - \int \sec\theta \, (\sec^2\theta - 1) d\theta$

$= \sec\theta\tan\theta - I + \int \sec\theta \, d\theta$

$$2I = \sec\theta\tan\theta + \int \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta} d\theta = \sec\theta\tan\theta + \int \frac{d(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta}$$

$$I = \frac{1}{2}[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|] + C$$

$$L = \frac{1}{4}[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|]_0^{\tan^{-1}2} = \frac{1}{4}\left[\sqrt{5} \times 2 + \ln\left|\sqrt{5} + 2\right|\right]$$

$$= \frac{1}{4}\left[\sqrt{5} \times 2 + \ln\left|\sqrt{5} + 2\right|\right] \approx \underline{\underline{\mathbf{1.4789428575446}}}$$

Method 2

$$L = \int_0^1 \sqrt{1 + 4x^2} \, dx$$

Put $2x = \sinh\theta \Longrightarrow dx = \frac{1}{2}\cosh\theta \, d\theta$

$\sqrt{1 + 4x^2} = \sqrt{1 + \sinh^2\theta} = \cosh\theta$

$$L=\int_0^{\sinh^{-1}2}\cosh\theta\left(\frac{1}{2}\cosh\theta\,d\theta\right)=\frac{1}{2}\int_0^{\sinh^{-1}2}\cosh^2\theta\ d\,\theta=\frac{1}{2}\int_0^{\sinh^{-1}2}\frac{1+\cosh 2\theta}{2}d\,\theta$$

$$=\frac{1}{4}\left[\theta+\frac{\sinh 2\theta}{2}\right]\begin{matrix}\sinh^{-1}2\\0\end{matrix}=\frac{1}{4}[\theta+\sinh\theta\ \cosh\theta]\begin{matrix}\sinh^{-1}2\\0\end{matrix}=\frac{1}{4}\left[\sinh^{-1}2+2\sqrt{5}\right]$$

$$\approx \underline{\underline{\mathbf{1.4789428575446}}}$$

Method 3

We calculate a standard integral first $\int\sqrt{1+x^2}\ dx$:

Let $y^2=1+x^2\Longrightarrow ydy=xdx$

Consider $I=\int\sqrt{1+x^2}\ dx=\int ydx=xy-\int xdy$

Hence $I=\frac{1}{2}[xy+\int(ydx-xdy)]=\frac{1}{2}\left[xy+\int\frac{(x+y)(ydx-xdy)}{x+y}\right]=\frac{1}{2}\left[xy+\int\frac{xydx+y^2dx-x^2dy-xydy}{x+y}\right]$

$$=\frac{1}{2}\left[xy+\int\frac{y(ydy)+y^2dx-x^2dy-x(xdx)}{x+y}\right]=\frac{1}{2}\left[xy+\int\frac{y(ydy)+y^2dx-x^2dy-x(xdx)}{x+y}\right]$$

$$=\frac{1}{2}\left[xy+\int\frac{(y^2-x^2)dx+(y^2-x^2)dy}{x+y}\right]=\frac{1}{2}\left[xy+\int\frac{dx+dy}{x+y}\right]=\frac{1}{2}[xy+\ln|x+y|]+C$$

$$=\frac{1}{2}\left[x\sqrt{1+x^2}+\ln\left|x+\sqrt{1+x^2}\right|\right]+C$$

$$L=\int_0^1\sqrt{1+4x^2}\,dx=\frac{1}{2}\int_0^1\sqrt{1+(2x)^2}\,d(2x)=\frac{1}{2}\left\{\frac{1}{2}\left[(2x)\sqrt{1+(2x)^2}+\ln\left|(2x)+\sqrt{1+(2x)^2}\right|\right]\right\}\begin{matrix}1\\0\end{matrix}$$

$$=\frac{1}{4}\left[\sqrt{5}\times 2+\ln\left|\sqrt{5}+2\right|\right]\approx\underline{\underline{\mathbf{1.4789428575446}}}$$

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27/5/2019